Macroscopic car condensation in a parking garage

Meesoon Ha and Marcel den Nijs

Department of Physics, University of Washington, Seattle, Washington 98195

An asymmetric exclusion process type process, where cars move forward along a closed road that starts and terminates at a parking garage, displays dynamic phase transitions into two types of condensate phases where the garage becomes macroscopically occupied. The total car density ρ_o and the exit probability α from the garage are the two control parameters. At the transition, the number of parked cars N_p diverges in both cases, with the length of the road N_s, as $N_p \sim N_s^{y_p}$ with $y_p = 1/2$. Towards the transition, the number of parked cars vanishes as $N_p \sim \epsilon^\beta$ with $\beta = 1$, $\epsilon = |\alpha - \alpha*|$ or $\epsilon = |\rho_o^* - \rho_o|$ being the distance from the transition. The transition into the normal phase represents also the onset of transmission of information through the garage. This gives rise to unusual parked car autocorrelations and car density profiles near the garage, which depend strongly on the group velocity of the fluctuations along the road.

I. INTRODUCTION

The scaling properties of nonequilibrium-type phase transitions are a topic of intensive current research [1–4]. From the theoretical side, focus has been mostly on model calculations using stochastic dynamics with simple local rules and local interactions. These models serve as prototypes for various physical processes, such as surface catalysis [5], population growth [6], surface growth [2,7], electronic transport in wires [8], traffic flow [9], and avalanches in granular materials [10]. The simplicity of the models is justifiable when, as we expect, the scale invariant collective fluctuations at large length scales are universal and insensitive to most microscopic details.

Queuing phenomena, like traffic jams behind slow moving trucks on single-lane highways, are common to many nonequilibrium processes. One of the interesting phenomena in such systems is the transition from a finite queue to an infinitely long one, i.e., the transition from a queue that does not scale with the road length N_s to one whose length is proportional to N_s. Such transitions can be induced by changing the total density of cars in the system or by varying the probability rate at which cars pass the truck [11].

A simplification of the above is first to replace the slow moving truck by a stationary object, and then to ignore the excluded volume effects in the queue by allowing the cars behind this stationary impurity to pile on top of each other. This bare-bone version of the phenomenon can then be rephrased in terms of the parking garage model introduced in this paper. An asymmetric exclusion process (AEP) on a road starting and terminating at a garage with an infinite parking capacity. The control parameters are the total number of cars, N_c, in the system (which compared to the road length N_s, defines a total car density $\rho_o = N_c/N_s$) and the probability α with which cars exit the garage (compared to the hopping probability along the road, set equal to one). The infinitely long queue is represented by a macroscopic occupation of the garage. The density of parked cars, $\rho_p = N_p/N_s$, is nonzero when the number of cars at the garage N_p is proportional to N_s.

The analogy with equilibrium Bose condensation is obvi-

ous as well as the basic question, i.e., whether the scaling properties of such queuing transitions are universal. For example, how sensitive are the exponents β and x_p in

$$\rho_p \sim |\alpha - \alpha_c|^\beta \quad \text{for} \quad \alpha < \alpha_c \tag{1}$$

and

$$\rho_p \sim N_s^{-x_\rho} \quad \text{at} \quad \alpha_c \tag{2}$$

to the details of the queuing dynamics, particularly to the simplifications we made. As a start, we need to establish and understand the scaling properties of the most simplest model in as much detail as possible. The scaling properties of the parked car condensation transitions presented here are indeed already surprisingly rich. After this, we will be well positioned to restore the omitted queuing details one-by-one, and establish how robust those scaling properties are.

This paper is organized as follows. The parking garage model is introduced in Sec. II, and the phase diagram is presented in Sec. III. In Sec. IV, we explore the relation to Kardar-Parisi-Zhang (KPZ) type surface growth. The group velocity is introduced in Sec. V, and its important role in setting the density profile in the various phases is reviewed in Sec. VI. In Sec. VII, we do the same for the density-density correlations. Sections VIII and IX form the core of this paper. We present our numerical Monte Carlo analysis for the parked car density at the condensate-normal phase transitions in Sec. VIII, together with explanations of these results. Next, we do the same for the parked car fluctuations in various phases and at the phase boundaries in Sec. IX. Finally, in Sec. X, we summarize our results.

II. PARKING GARAGE MODEL

Consider a one-dimensional road of N_s sites with periodic boundary conditions (PBC). Each site on this road, $1 \leq x \leq N_s - 1$, is either empty or occupied by only one car, $n_x = 0,1$. The parking garage at site N_s has no occupation upper limit, $N_p \equiv n_{N_s} = 0,1,2,3, \ldots$. The total number of cars (N_c) in the systems is conserved. From the perspective of other
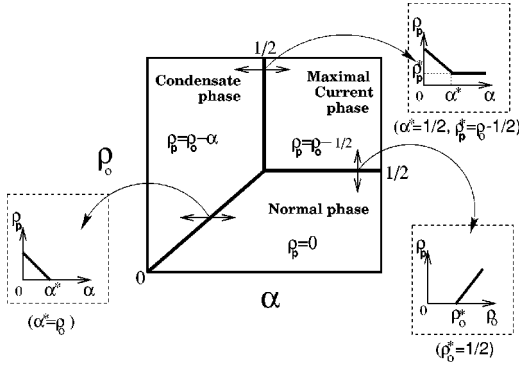
FIG. 1. Phase diagram with the total car density ρ_o and the exit probability α from the garage. At the phase transition lines into the N phase, the parked car density ρ_p vanishes, and the garage ceases to be macroscopically occupied (shown schematically in the insets).

physical processes, the cars can be reinterpreted just as well as, e.g., molecules (driven down a circular tube), kinks in steps on crystal surfaces, electrons driven around a wire by an electric field, or domain walls in magnetic spin chains.

The stochastic update rule, from time t to $t+1$ is sequential. One of the sites, $1 \leq x \leq N_s$, is selected at random with uniform probability $p(x) = 1/N_s$. If that site is part of the road, $1 \leq x \leq N_s - 2$, and occupied, the car moves forward to over one unit, presuming site $x+1$ is empty; otherwise nothing happens. If the selected site is the one just in front of the garage, $N_s - 1$, and is occupied, then that car moves with certainty into the garage, irrespective of the occupation level of the garage. If the chosen site is the garage, $x = N_s$, the probability for a car to jump out of it onto site $x = 1$ is equal to α, independent of the number of cars at the garage, provided that there is at least one available and site $x = 1$ is empty. This exit probability from the garage is smaller than the hopping probability on the road, $0 \leq \alpha \leq 1$.

The above process has two control parameters: the total density of cars ρ_o and the probability α with which cars can escape from the garage. We define the following quantities: the total car density $\rho_o = N_c/N_s$, the parked car density $\rho_p = N_p/N_s$, and the on-the-road density $\rho_r = \rho_o - \rho_p$. Only ρ_p and ρ_r fluctuate.

The parking garage is the new aspect to this otherwise well-studied AEP. The latter is exactly soluble by the Bethe ansatz for periodic boundary conditions (a closed loop road without garage) [12], and the stationary state properties are exactly known for an open road with reservoirs on both sides [14]. We will use and comment on these different setups and exact results in the next following sections.

III. THE PHASE DIAGRAM

Figure 1 shows our phase diagram. It contains three phases. In two of them the garage is macroscopically occupied: the condensate (C) phase (where the garage controls the density of cars on the road), and the maximal current (MC) phase (where the road capacity controls the flow). In the normal (N) phase, the garage contains a finite number of parked cars (typically only a few). This structure can be de-

duced almost completely from earlier results for AEPs without the garage, such as the AEP with closed periodic boundary conditions (no garage), i.e., KPZ growth [12,13] and the AEP with open boundary conditions hooked-up to reservoirs on both ends [14,15].

In the two condensate phases, C and MC, the garage acts like a reservoir, and the model reduces to the open road version studied by Derrida *et al.* [14] and others [15]. In their version, the road is in contact with car reservoirs at both ends, $x = 1$ and $x = N_s - 1$, such that a car jumps onto $x = 1$ with probability α (if empty) and leaves from $x = N_s - 1$ with probability β (if occupied). In our model β is always equal to 1.

A dynamic second-order phase transition takes place between the MC phase (where the road density is at a constant value, $\rho_r = \frac{1}{2}$) and the C phase (where the road density $\rho_r = \alpha$ varies with the inlet probability α and this result is already correct within mean field theory [14,15]). The MC phase appears because raising the density any further would reduce the flow efficiency (due to overcrowding). This phenomenon has been canonized recently into a "maximal current principle" [16], which states that the bulk road density takes the value that maximizes the flow.

Reduce the total number of cars inside the C and MC phases, i.e., walk in phase diagram towards the N phase along a line of constant α. This has no effect on the cars on the road and their fluctuations because all removed cars are taken from the parked ones residing inside the garage. These surplus cars are dynamically inert. This continues until the reservoir is depleted, the point at which the transition to the N phase takes place. The road density becomes equal to the total car density $\rho_r = \rho_o$. From the C (input-limited) phase perspective, this happens at $\rho_o = \alpha$, see Fig. 1, because $\rho_r = \alpha$ inside the C phase [14–16]. From the MC (road-limited) phase perspective, it occurs at $\rho_o = \frac{1}{2}$ because $\rho_r = \frac{1}{2}$ inside MC. Therefore, the phase boundaries into the N phase are located at $\rho_o = \alpha$ and $\rho_o = \frac{1}{2}$.

The next issue is how many types of normal phases exist in the lower right side of the phase diagram. In an N phase, the garage acts like a localized impurity (a blocking-type site), and it contains typically only a few cars. Suppose we put a cap N_m on the occupation of the garage, $N_p = 0, 1, \ldots, N_m$. This cannot affect the properties of the model in the N phase, except very close to the transition into the condensate phases (where the number of parked cars N_p and the fluctuations in N_p diverge). Janowsky and Lebowitz [11] (JL) studied such a capped version of the AEP model, a closed loop road with periodic boundary conditions, without a garage, but with one special bond where the hopping probability is reduced from 1 to α. This is equivalent to setting the occupation limit of the garage to $N_m = 1$. Their phase diagram has the same control parameters as ours. It has particle-hole symmetry with respect to $\rho_o = \frac{1}{2}$, see Fig. 2. The normal phase in the lower right hand corner is similar to our model. The second normal phase in the upper right corner is equivalent to it by particle-hole symmetry, and the intermediate area is a coexistence region, the jammed phase. In the low-density N phase, the cars are uniformly distributed, but at $\rho_o = \alpha/(1 + \alpha)$ a traffic-jam-type shock wave de-
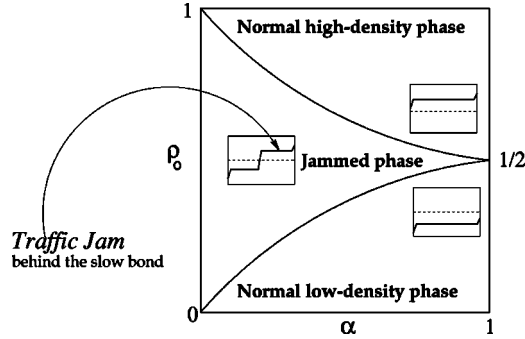
FIG. 2. Phase diagram of the Janowsky-Lebowitz model: an AEP with periodic boundary conditions and a slow bond. The corresponding density profiles are schematically shown in the insets.

velops between low- and high-density N-type phase regions.

This jammed phase is reminiscent of the condensate phase (macroscopic occupation of the parking garage) in our model. A shock wave does not appear in our model because the garage absorbs the queue. It is also clear that if we would interpolate between the JL model and our model by slowly increasing $N_m = 1 \rightarrow \infty$, the queuing transition would be delayed and would be shifting smoothly, with our N-C phase boundary as the limiting case. JL focused on the finite-size-scaling (FSS) properties of the fluctuations in the position of the tail of the shock wave, $\Delta l \sim L^\gamma$, and found $\gamma = \frac{1}{2}$ (except for $\rho_o = \frac{1}{2}$ where a cancellation of the leading time of flight-type fluctuations gives rise to $\gamma = \frac{1}{3}$) [11]. Monitoring the location of the shock wave is analogous to observing the parked car density fluctuations in our model inside the C and MC phases. JL did not address the scaling properties at the queuing transition itself, but their transition is definitely second order (the location of the shock wave moves continuously with the total density away from the blockage point).

IV. KPZ GROWTH

The AEP can be interpreted as a lattice representation of the Burgers equation with a discretized velocity field, both in location and in its values, $n(x) = 0, 1$. The latter is equivalent to the so-called body-centered solid-on-solid growth model [7,17], a lattice realization of KPZ-type surface growth. Each occupied site represents a down step along the one-dimensional (1D) surface and every empty site an up step. The surface heights $h(x + \frac{1}{2})$ are limited to even/odd values at even/odd lattice sites, $h(x + \frac{1}{2}) - h(x - \frac{1}{2}) = 1 - 2n(x)$. The surface looks like the alternatively stacked bricks of a masonary wall, but rotated over 90 deg. Each car jump to the right represents the deposition of a new vertical 1×2 brick. The average car density on the road represents the average slope of the surface. At half density the surface is not tilted.

Periodic boundary conditions are the most common in theoretical studies of surface growth, and this brick deposition version of KPZ growth is exactly soluble by the Bethe ansatz [12]. It is a special line in the general phase diagram of the one-dimensional well-known XXZ spin-$\frac{1}{2}$ quantum spin chain Hamiltonian, and also in the so-called six-vertex model of 2D equilibrium critical phenomena [18].

From the surface growth perspective, the open road setup with reservoirs represents an open ended 1D interface with modified particle deposition probabilities α and β at its end points. In the MC phase, where the car density is locked to $\rho_o = \frac{1}{2}$, the crystal surface maintains a net zero tilt. For $\alpha < \frac{1}{2}$ and/or $\beta < \frac{1}{2}$, the reduced growth probabilities at the edge create a nonzero globally tilted surface. Moreover, along the line $\alpha = \beta < \frac{1}{2}$, two tilts coexist. Crossing that line amounts to undergoing a first-order phase transition with different tilt angles. The lock-in transition from the C phase into the MC phase represents a second-order dynamic facetting transition.

Our parking garage setup translates into a surface growth dynamics realization with periodic boundary conditions and a localized defect. The garage could represent something like a stacking defect, where the step height is unlimited (and somehow energetically cost free). The cliff height at the stacking fault can be microscopic (the N phase) or macroscopic (the C and MC phases). This depends on the modified growth probability α near the defect (on top of the cliff, somewhat similar to a Schwoebel energy barrier) and also depends on the global net tilt angle (the total car density). In the MC (C) phase, the average slope of the surface, excluding the cliff, is zero (nonzero).

V. CORRELATIONS AND GROUP VELOCITY

It is well known and easy to show that the stationary state of KPZ growth with PBC, the closed loop road without obstacles where $\alpha = 1$ and $n(N_s) = 0, 1$, is completely random without any correlations whatsoever between finding a car at sites i and j for any $i \neq j$ including nearest neighbor sites. In the surface representation, this means that up and down steps are placed at random [17] and that the width of the interface $W \sim N_s^\chi$ scales with the random walk exponent $\chi = \frac{1}{2}$.

The stationary state remains uncorrelated in our model as well (as shown below) except near the garage. Those correlations are governed by the dynamic scaling exponent and the group velocity with which fluctuations travel along the road.

In 1D KPZ-type surface growth, all characteristic times scale as $t \sim N_s^z$ with $z = 3/2$. In car language, this means that local car density fluctuations with spatial width l broaden in time as $l \sim t^{1/z}$. The value of the dynamic exponent z follows from the identity $z + \chi = 2$ implied by Galilean invariance of the Burgers equation [12,13], together with the disordered $\chi = \frac{1}{2}$ nature of the stationary state. The exact (Bethe ansatz) solution of this specific model confirms this general result.

Fluctuations travel with a group velocity $v_g = 1 - 2\rho_r$ to the right (towards the garage) along the road. From the KPZ surface growth perspective, the group velocity represents the average slope of the KPZ growing surface, and its fluctuations grow perpendicular to the surface orientation. From the AEP formulation perspective, the value of v_g is set by the average stationary state current $j = \rho_r(1 - \rho_r)$ (easy to derive since the stationary state is Gaussian) and the definition of the group velocity is

$$v_g = \frac{\partial}{\partial \rho_r} j(\rho_r) = (1 - 2\rho_r). \tag{3}$$

This v_g played an important role, e.g., in the AEP study by Majumdar and Barma [19] of tagged-particle diffusion, and in describing phase transitions between steady states in the open road setup [20].

VI. DENSITY PROFILES IN THE VARIOUS PHASES

The on-the-road car density profiles in the C and MC phases are known exactly from the open boundary model studies, in particular the one by Derrida *et al.* [14]. These profiles play an important role in our later discussions and it is useful to review how the group velocity enters in qualitative explanations of these exact results.

In the road-limited MC phase at $\alpha > \frac{1}{2}$, the density tail has a power-law shape,

$$\rho(x) \simeq \rho_r + A_e x^{-\nu}, \tag{4}$$

at both edges $e = L(\text{left}), R(\text{right})$, with x the distance from the edge of the road (garage entrance in our model) and $\rho_r = \frac{1}{2}$ the on-the-road car density. In contrast, in the input-limited C phase at $\alpha < \frac{1}{2}$, the car density is constant, $\rho(x) = \rho_r$, at the beginning of the road (garage exit), and has an exponential tail at the end of road (garage entrance),

$$\rho(N_s - x) \simeq \rho_r + A x^{-3/2} e^{-x/\xi}, \tag{5}$$

with the on-the-road car density $\rho_r = \alpha$ and the correlation length $\xi = -1/\ln[4\alpha(1-\alpha)]$.

Power laws with exponent $\nu = \frac{1}{2}$ arise naturally in this problem because of its critical fluctuations. The bulk properties of the KPZ stationary state are invariant under a rescaling of all lengths as $x' = bx$, all times as $t' = b^z t$, and the surface heights as $h' = b^\chi h$. The car density scales, therefore, naively as $\rho = \partial h / \partial x \sim b^{\chi - 1}$. This means that power-law tails in the density distribution near the edge of the road, with exponent $\chi - 1 = -\frac{1}{2}$, like in Eq. (4), are to be expected.

On the other hand, the disordered nature of the stationary state suggests exponential profiles, like in Eq. (5). Indeed, in the bulk, the stationary state density-density correlation function

$$g(r) = \langle \rho(x+r)\rho(x) \rangle - \langle \rho(x+r) \rangle \langle \rho(x) \rangle \tag{6}$$

does not decay as an r^{-1} power law as may be suggested by the above scaling argument, but instead decays exponentially with a very short correlation length, that is zero in this specific model since the cars on the road are totally uncorrelated in the stationary state.

To make sense of Eqs. (4) and (5), it is important to realize that the density profile near the edge of the road incorporates temporal correlations, and also to appreciate the role of the group velocity.

The power-law profile in the MC phase is the result of correlations with cars that reached the edge of the road and moved into the garage at earlier times. Such correlations spread in time over a spatial distance $l \sim t^{1/z}$. The group ve-
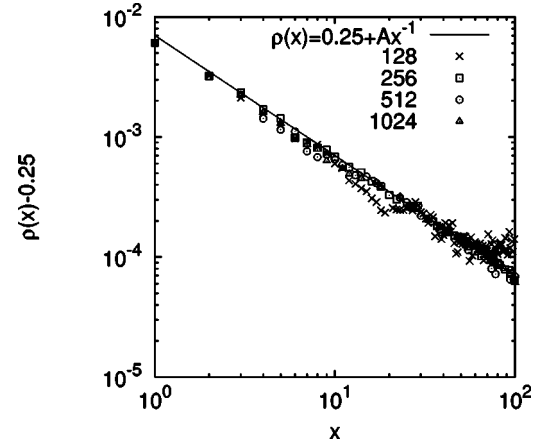

FIG. 3. Double logarithmic plots of the density profiles in the N phase ($\rho_o = 0.25$ and $\alpha = 1$), showing a $1/x$ tail at the beginning of the road (garage exit).

locity is zero in the MC phase and therefore the cars at a distance l from the edge of the road are (power-law) correlated with those that entered the garage at times earlier than $\Delta t = l^z$. They have no knowledge about cars arriving at the garage more recently. This explains the power-law tail in the density distribution.

In the C phase, the same correlation spreading takes place within a moving frame of reference with nonzero group velocity. Information reaches the edge of the road (garage entrance) at a rate v_g (linear in time) faster than it can spread backwards as $l \sim t^{1/z}$ (since $1/z = 2/3 < 1$). Memory has no opportunity to develop near the edge of the road, and the density profile adjusts itself at the road end as if the cars on the road are completely uncorrelated. The exponential tail reflects a suction-type effect; the excluded volume limitation on the car mobility does not act on the cars departing from the road.

Similarly, in the MC phase, power-law correlations with earlier cars that escaped from the garage and entered the road give rise to a power-law tail in the density profile at the beginning of the road (near the exit of the garage), while in the C phase, this information travels faster away from the road start than it can spread backwards. So, new cars entering the road do so completely uncorrelated, resulting in no road-start tail in the profile whatsoever. This confirms why the MC phase is road limited and the C phase is input limited. In the latter, the car supply at the garage controls the density near the edge of the road and also everywhere else on the road.

In the N phase, we find numerically, from Monte Carlo simulations, an exponential tail at the road end and an $1/x$ tail at the road start as shown in Fig. 3. The group velocity is also nonzero in the N phase, which explains the exponential exit tail as follows. Just like in the C phase, the garage is invisible to the incoming of cars from the road; fluctuations arrive at the garage faster (at constant velocity) than they can spread backwards (with $\Delta l \sim t^{1/z}$). The fact that in the N phase only a few cars reside inside the garage is invisible to the cars entering the garage.

A $1/x$ tail at the beginning of the road is what one expects from the deterministic part of the Burgers equation. The solution of the deterministic Burgers equation,

$$\frac{\partial v}{\partial t} + \lambda v \frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}, \tag{7}$$

with the velocity v pinned at a specific nonzero value at site $x=0$, is readily seen (e.g., in the Hopf transformed formulation) to be of the generic form $v(x) \sim 1/(1+ax)$, i.e., having the $1/x$ power-law shape.

In the C phase the system selects the value $a=0$ for the constant, and in the N phase, $a \neq 0$. In both cases, the group velocity $v_g > 0$ carries away KPZ fluctuations faster than they can spread, such that the KPZ noise term can be ignored at the road start (stationary frame of reference). The only noise that remains is that from the random process by which cars are being taken out of the garage and put on the road. In the C phase, the supply of cars is bottomless ($\rho_p > 0$), such that $a=0$ (the entire on the on-the-road bulk car density is ruled by the garage), while, in the N phase, the supply of cars is limited and the garage does not overwhelm the road, such that $a \neq 0$.

VII. DENSITY-DENSITY CORRELATIONS

It is useful to discuss how the group velocity affects the car-car correlation function,

$$g(r,\tau) = \langle \rho(x+r)_{t+\tau} \rho(x)_t \rangle - \langle \rho(x+r)_{t+\tau} \rangle \langle \rho(x)_t \rangle. \tag{8}$$

According to scaling theory, it obeys the form

$$g(r,\tau) = b^{2(\chi-1)} g(b^{-1}r, b^{-z}\tau) = \tau^{2(\chi-1)/z} F\left(\frac{r}{\tau^{1/z}}\right). \tag{9}$$

In the limit $\phi = r/\tau^{1/z} \to 0$, the scaling function $F(\phi)$ must approach a constant since the autocorrelation decays as a power law,

$$g(0,\tau) \sim \tau^{2(\chi-1)/z} \sim \tau^{-2/3}, \tag{10}$$

with $\chi = \frac{1}{2}$. In the opposite limit, of large ϕ, the scaling function must decay exponentially because $g(r,0)=0$ due to the random nature of the stationary state (or, more generically, it decays exponentially with some correlation range ϕ_o of the same order as the interaction range between cars). The above scaling relation suggests the form

$$F(\phi) \sim \phi^{2(\chi-1)} e^{-\phi/\phi_o} \tag{11}$$

in the limit $\phi = r/\tau^{1/z} \to \infty$, such that

$$g(r,\tau) \sim r^{2(\chi-1)} e^{-r/\phi_o \tau^{(1/z)}} \sim \frac{1}{r} e^{-r/l} \tag{12}$$

with $\chi = 1/2$ and $l = \phi_o \tau^{1/z}$. The length l in the exponential defines the "correlation cone" that we already mentioned above. Within the cone the correlations decay as a power law, and outside it the cars are uncorrelated (as in the sta-
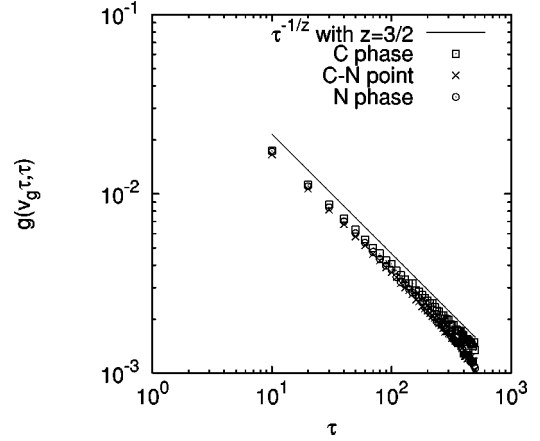

FIG. 4. Double logarithmic plots of the density-density correlation function $g(r,\tau)$ in the moving frame of reference, i.e., with $r = v_g \tau$ and $v_g = (1-2\rho_r)$. The C phase data are obtained at $\rho_o = 0.75$ and $\alpha = 0.25$, the C-N point data at $\rho_o = \alpha = 0.25$, and the N phase data at $\rho_o = 0.25$ and $\alpha = 1$.

tionary state). The above discussion ignores finite-size effects. For open road or garage type boundary conditions, the correlator explicitly depends on the initial position x_0 of the car at time t_0 and also on the road length N_s,

$$g(r,\tau;x_0,N_s) = b^{2(\chi-1)} g(b^{-1}r, b^{-z}\tau; b^{-1}x_0, b^{-1}N_s). \tag{13}$$

Finite-size effects set in at times when the correlation cones hit the road edges, $\tau \sim (x_0)^z$ or $\tau \sim (N_s - x_0)^z$. At times longer than $t_0 \sim N_s^z$, we therefore expect that the car-car correlator decays exponentially, e.g., as

$$g\left(0,\tau;\frac{1}{2}N_s,N_s\right) \sim N_s^{2(\chi-1)} e^{-a(\tau/N_s^{1/z})}. \tag{14}$$

The above analysis applies to the MC phase, where the drift (group) velocity is zero. In the C phase (and also the N phase), we need to switch to the moving frame of reference by replacing $r \to r + v_g \tau$. This has some peculiar consequences. For example, the autocorrelation function $g(0,\tau)$ decays exponentially in time (even at short times). The correlation cone l is slanted in the direction of the flow (the correlations move with the flow) and l widens slower than linear in time, such that the $r=0$ line in the world sheet lies outside the correlation cone. The KPZ-type correlations are somewhat hidden. In order to expose them, one needs to plot them in some special manner, e.g., $g(v_g \tau, \tau)$ (the autocorrelator in the moving frame of reference), as illustrated in Fig. 4.

VIII. SCALING AT AND NEAR THE CONDENSATION TRANSITIONS

Let us turn now to the scaling properties of the two condensate phase transitions from the C and MC phases into the N phase. The stationary state value of the parked car density $\rho_p = N_p/N_s$ acts as the order parameter. We expect it to obey the FSS relation
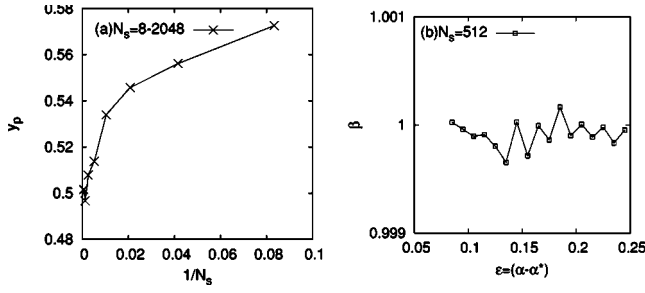
FIG. 5. The parked car (order parameter) exponents (a) y_p = 0.505(5) and (b) β = 1.000(1), defined as $N_p \sim N_s^{y_p}$ and $\rho_p \sim (\alpha - \alpha^*)^\beta$, at $\rho_o = \alpha = 0.25$ of the *C-N* phase boundary.

$$\rho_p(\epsilon, N_s^{-1}) = b^{-x_p} \rho_p(b^{y_\epsilon} \epsilon, b N_s^{-1}) \qquad (15)$$

near the two condensation transition lines with a scaling factor b, and $\epsilon = (\alpha - \alpha^*)$ or $\epsilon = (\rho_o^* - \rho_o)$, a measure of the distance from the transition.

In the *N* phase, the density of parked cars is zero, $\rho_p = 0$. When approached from the *C* and MC sides, ρ_p goes to zero as $\rho_p \sim |\epsilon|^\beta$, where $\beta = x_p / y_\epsilon$. The removal of cars from the road is accommodated by taking them from the passive inert ones residing inside the garage. This implies that ρ_p must vanishes linearly at the transition, and that $x_p = y_\epsilon$. Our numerical simulations confirm that $\rho_p \sim |\epsilon|$, as $\beta = 1$ in Figs. 5 and 6.

The total number of parked cars (N_p) scales as

$$N_p(\epsilon, N_s^{-1}) = b^{y_p} N_p(b^{y_\epsilon} \epsilon, b N_s^{-1}), \qquad (16)$$

with exponent $y_p = 1 - x_p$ according to Eq. (16) and $N_p = \rho_p N_s$. In the *N* phase, the density ρ_p remains zero, while the total number of parked cars (N_p) diverges towards the transition as $N_p \sim |\epsilon|^{(1 - x_p)/y_\epsilon}$. We find numerically that this power law is linear as well, $N_p \sim |\epsilon|$. Combined with the linear scaling of the car density from the *C* or MC side (implying $y_p = x_p$), this yields $y_p = y_\epsilon = \frac{1}{2}$. The exponent y_p determines the FSS behavior of the total number of parked cars at the transition point itself, $N_p \sim N_s^{y_p}$ (and also that the density of parked cars vanishes as $\rho_p \sim N_s^{y_p - 1}$).
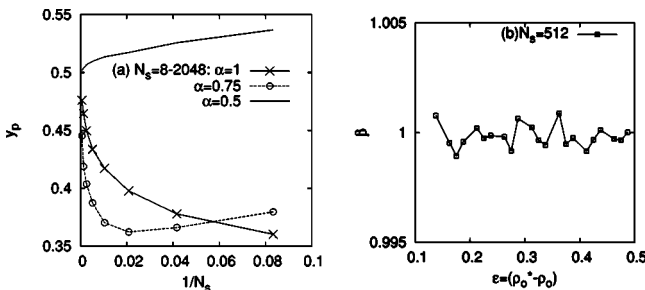

FIG. 6. The parked car (order parameter) exponents (a) y_p = 0.495(5) and (b) β = 0.997(3), at $\rho_o = 1/2$ and $\alpha = 1$ of the *N*-MC phase boundary. For the other two curves in (a), taken along the MC-*N* phase boundary at $\alpha \geq 1/2$, the exponent y_p retains the same value, but is subject to large/small corrections to finite-size scaling originating from 2/3 power-law density profile at the beginning of the road (garage exit).
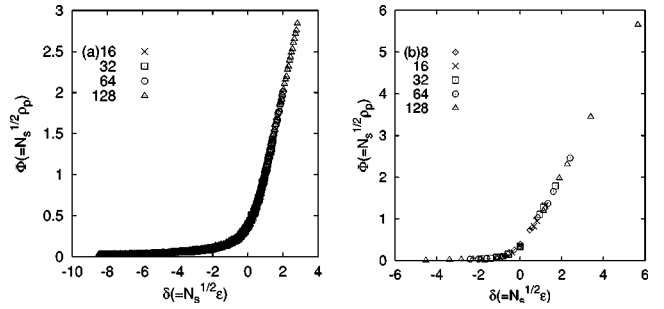

FIG. 7. Scaling function $\Phi(\xi)$ defined in Eq. (17) at the same transition points as in the previous two figures: (a) $\rho_o = \alpha$ = 0.25 (*C-N*) and (b) $\rho_o = 1/2$, $\alpha = 1$ (MC-*N*). The numerical data collapse very well for $x_p = y_\epsilon = 1/2$, as suggested in Eq. (17). The scaling function $\Phi(\xi) \to 0$ for $\xi \ll 0$ and $\Phi(\xi) \to \xi$ for $\xi \gg 0$.

Figure 5 shows the numerical results of y_p and β at point $\alpha = 0.25$ of the *C-N* phase boundary and Fig. 6 at $\alpha = 1$ (and 0.75) of the MC-*N* phase boundary. Notice that the FSS corrections in the exponent y_p are much stronger at the MC-*N* transition than at the *C-N* transition.

The validity of the scaling relations is further illustrated by plotting the scaling function $\Phi(\xi)$ in the following form:

$$\rho_p(\epsilon, N_s^{-1}) = N_s^{-x_p} \Phi(N_s^{y_\epsilon} \epsilon), \qquad (17)$$

and the associated data collapse while moving through the transition points at constant ρ_o and constant α, respectively. The curves in Fig. 7 collapse very well.

The fluctuations and the FSS corrections to the number of parked cars are a mirror of the density-density correlations and the density profile of cars on the road. They also reflect how the latter builds up as a function of the length of the road. At the transition points, the bulk value of the on-the-road density $\rho_r = N_r / N_s$ is exactly equal to the total number of cars in the system divided by the road length, $\rho_r = N_c / N_s$, such that there would be no need for any car to remain inside the garage.

We find numerically that at the *C-N* transition point, the density profile retains the same structure as inside the *C* phase; with no tail at the beginning of the road (garage exit) and an exponential tail at the end of the road (garage entrance). Such profiles cannot account for the $y_p = \frac{1}{2}$ FSS divergence in the number of parked cars. The scaling behavior $N_p \sim N_s^{y_p}$ must, therefore, reflect directly the corrections to FSS in the bulk density of cars on the road. The value $x_p = \frac{1}{2}$ naturally arises because at the transition point, $\rho_p \sim \rho_r$, and $\rho_r = \partial h / \partial x$ scales as $L^{x-1} \sim L^{-x_p}$, where L corresponds to a given length of the road N_s.

A more intuitive explanation follows again from the nonzero group velocity at the *C-N* phase boundary. As mentioned above in the discussion of the density distribution tail in the *C* phase, the events by which cars enter the road from the garage are completely uncorrelated due to the slanting of the correlation cones (the correlations move with the flow towards the garage and spread slower than linearly, only as $l \sim t^{1/z}$, such that communications with later events at the beginning of the road are impossible). So the entry events to
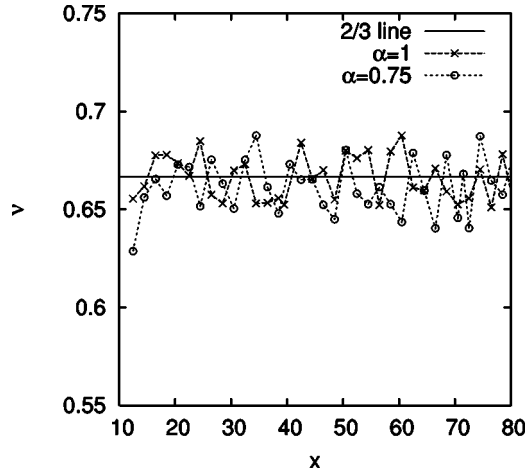
FIG. 8. Effective (finite-size-scaling type) value for the exponent ν of the density profiles at the beginning of the road: $\nu = 0.66(1)$ at $\rho_o = 1/2$ and $\alpha = 1$ (or 0.75). At the MC-N phase boundary, the density profiles decay as a 2/3 power law, instead of the 1/2 power law found inside the MC phase.

the road from the garage behave like uncorrelated random noise, and the fluctuations in the number of cars scale, therefore, as the square root of time. The time in question is proportional to N_s because the fluctuations (created at the garage exit) move along the road with velocity v_g, and are wiped out after they return to the garage. From this, it follows that the fluctuations in the number of parked cars scales as $\sqrt{N_s}$. Moreover, N_p cannot be negative, which means that the fluctuations sample the bottom of the garage and that the transition from the C to N phase, therefore, takes place when the garage contains $N_p \sim \sqrt{N_s}$ cars.

In summary, the scaling at the C-N phase boundary is governed by the bulk fluctuations in the-on-the-road density, which is ruled by the nonzero group velocity of KPZ fluctuations, and this leads directly to random noise, like $N_p \sim \sqrt{N_s}$, corrections to FSS.

The FSS corrections to scaling in exponent $x_p(y_p)$ at the MC-N phase boundary are much more complex. The (bulk) group velocity is zero, and power-law density profiles are realized at both edges of the road. At the end of the road, the density profile follows a critical exponent $\nu = \frac{1}{2}$, the same power as that inside the MC phase discussed in Sec. VI. However, at the road start (the exit of the garage), the power-law exponent changes from $\nu = \frac{1}{2}$ inside the MC phase to $\nu = \frac{2}{3}$ at the MC-N transition. This is shown in Fig. 8 for $\alpha = 1$ and $\rho_o = \frac{1}{2}$, where the effect is the strongest.

The $\frac{2}{3}$ power law does not change the $N_s^{1/2}$ FSS behavior of the number of parked cars. It is responsible, however, for strong corrections to FSS, as clearly visible in Fig. 6(a). The $\nu = \frac{2}{3}$ power-law profile contributes only a subdominant term to ρ_p because it decays faster than the two $\nu = \frac{1}{2}$ contributions (from the density profile at the end of the road and from the KPZ-like bulk road density fluctuations). We have not achieved yet a good understanding of this novel value, $\frac{2}{3}$, for the exponent of the density profile at the beginning of the road. It obviously lies correctly in between the MC and N values, $\frac{1}{2}$ and 1, respectively. Moreover, its value, $\nu = \frac{2}{3}$, is

likely linked to the KPZ dynamic exponent $z = \frac{3}{2}$. But it remains unclear how to glue the following pieces together.

At the MC-N transition, the group velocity of the fluctuations, $v_g = 1 - 2\rho_r$, is still zero (but only barely) since $\rho_r = \frac{1}{2}$. This means that time of flight aspects, which dominate the C and N phases, do not come into play. Near the garage, the tail in the density profile, $\rho(x) = \frac{1}{2} + \Delta\rho(x)$, creates a local nonzero group velocity $v_g(x) = -2\Delta\rho(x)$ pointing back into the garage (instead of away from it as in the C and N phases).

Recall from Sec. VI that the density profile in the MC phase has the same type of power-law profile $\Delta\rho(x) \sim x^{-\nu}$, but with the ''KPZ'' (power-counting) exponent value $\nu = \frac{1}{2}$. In that case, fluctuations at the road start do not reach the bulk of the road: the leading edge of the information cone is stationary because the backward movement of its center of mass $x_c \sim -t^{1/(1+\nu)}$ (implied by $dx_c/dt \sim x_c^{-\nu}$) matches exactly the rate of its spreading, $x \sim t^{1/z}$. So the $\nu = \frac{1}{2}$ density profile fully screens the garage from view in the bulk of the road. Exactly the same screening of the information takes place at the end of the road (garage entrance) since also there the MC phase density profile has exponent $\nu = \frac{1}{2}$ (but in an opposite forward moving v_g sense and with a negative density profile amplitude).

The density profile exponent $\nu = \frac{2}{3}$ at the MC-N transition does not fully screen the garage any more from observers located far away on the road. Total screening is not needed because KPZ fluctuations start to tunnel through the garage since it has only a $\sqrt{N_s}$ occupation. It is yet unclear to us, however, how to deduce (in a convincing manner) the exponent $\nu = \frac{2}{3}$ from these considerations.

IX. PARKED CAR FLUCTUATIONS

In this section, we explore the fluctuations in the parked car density and also car-car correlations on the road. Of particular interest is the onset of transmission of information through the garage at the phase transitions. In the two condensate phases, the garage acts as a car reservoir and as a sink of fluctuations, while in the normal phase it contains only a few cars and transmits fluctuations.

The temporal fluctuations in the total number of parked cars, $G(N_s, \tau)$, measures also the fluctuations in the total number of cars on the road. It is therefore equal to the integrated car-car correlator (defined in Sec. VII),

$$G(N_s, \tau) = \sum_{x_1, x_0 = 1}^{N_s - 1} \left[\langle \rho(x_0, t_0) \rho(x_1, t_1) \rangle - \bar{\rho}(x_0) \bar{\rho}(x_1) \right]$$

$$= \sum_{x_0 = 1}^{N_s - 1} \sum_{r = -x_0 + 1}^{N_s - x_0 - 1} g(r, \tau; x_0, N_s), \qquad (18)$$

with $\tau = t_1 - t_0$ and $r = x_1 - x_0$. The summations run over all road sites x_0 and distances r that fit on the road. Direct integration of the scaling relation, Eq. (13), yields that (in the MC phase) G obeys the scaling form

$$G(N_s, \tau) = b^{2\chi} \, G(b^{-1} N_s, b^{-z} \tau) = N_s^{2\chi} \, \mathcal{F}(\tau/N_s^z). \qquad (19)$$
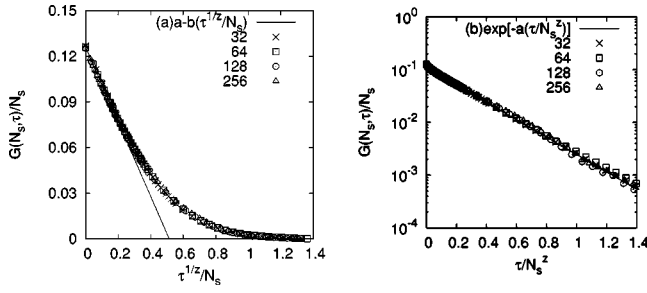
FIG. 9. The parked car correlator $G(N_s, \tau)$ in the MC phase: for small τ (a) it decays linearly as a function of $\tau^{1/z}/N_s$, and for large τ (b) it exponentially as a function of τ/N_s^z, where $z = 3/2$. The data are obtained at $\rho_o = 0.75$ and $\alpha = 1$.

In the KPZ representation, G is the global slope-slope autocorrelator, and at $\tau = 0$, reduces to the conventional definition of interface width (second moment of the height distribution). We will now discuss how G behaves in the various phases and at the phase transition points.

A. Inside the MC phase

The following intuitive discussion tells us how the scaling function $\mathcal{F}$ behaves in the MC phase. Each $g(r, \tau)$ has a correlation cone of size $l \sim \tau^{1/z}$. The cars within this cone are correlated with the one at site x_0 at time t_0 as $\tau^{2(\chi-1)/z}$. The integration over x_0 and r in Eq. (18) yields

$$G \sim N_s \times l \times \tau^{2(\chi-1)/z} \sim N_s \tau^{(2\chi-1)/z}, \qquad (20)$$

where the correlation cones $l \sim \tau^{1/z}$ are assumed to be small with respect to the road size N_s. The exponent is equal to $\chi = \frac{1}{2}$, such that $G \sim N_s$, and that $\mathcal{F}$ approaches a constant in the limit $\tau/N_s^z \to 0$.

This estimate fails to take into account finite-size effects. Correlations are truncated near the two road edges (all information is entering the garage). The loss term is of the order

$$2 \int_0^l dx_0 \; (l - x_0) \tau^{2(\chi-1)/z} \sim l^2 \; l^{2(\chi-1)} \sim \tau^{2\chi/z}. \qquad (21)$$

This suggests that G is of the form

$$G = N_s \left(a - b \frac{\tau^{1/z}}{N_s} + \cdots \right), \qquad (22)$$

with constants a and b, and suggests that the scaling function $\mathcal{F}(\phi)$ is analytic at short times in the parameter $\phi = \tau^{1/z}/N_s$ instead of $\phi' = \tau/N_s^z$. Our numerical results shown in Fig. 9(a) are consistent with this.

In the opposite limit, $\phi, \phi' \to \infty$, where time is large compared to the length of the road, all correlation cones are limited and equal to $l \simeq N_s$, and G behaves as in Eq. (14), such that

$$G \sim N_s \times l \times g\left(0, \tau; \frac{1}{2} N_s, N_s\right) \sim N_s e^{-a(\tau/N_s^z)}. \qquad (23)$$
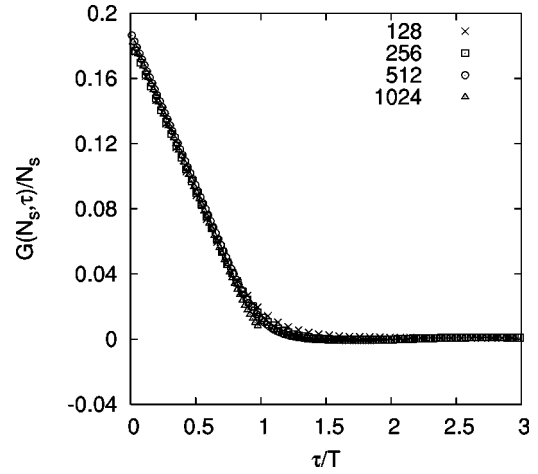

FIG. 10. The parked car correlator $G(N_s, \tau)$ in the C phase; it decays linearly as a function of τ and becomes zero after one time of flight $T = N_s/v_g (= \tau_{\text{flight}})$. Since it grows linearly with system size N_s, we plot G/N_s. The data are obtained at $\rho_o = 0.75$ and $\alpha = 0.25$.

Our numerical results in Fig. 9(b) are consistent with this as well.

B. Inside the C phase

In the C phase, the fluctuations scale with the same exponents as in the MC phase, but the nonzero group velocity $v_g = 1 - 2\rho_r$ changes completely the appearance of correlation functions, like $G(N_s, \tau)$. The density-density correlations spread just like in the MC phase, but only with respect to the moving frame of reference and with r replaced by $\tilde{r} = r + v_g \tau$. The correlation function $g(r, t)$ scales as a power law $g \sim \tau^{2(\chi-1)/z}$ at $\tilde{r} = v_g \tau$ (i.e., $r = 0$) but exponentially at nonzero r. The fluctuation cones, $l \sim t^{1/z}$, are slanted and move with the flow.

Figure 10 shows $G(N_s, \tau)$ inside the C phase as a function of time τ for various road sizes N_s. It decays linearly until hitting zero at $\tau_{\text{flight}} = N_s/v_g (\equiv T)$, and then it remains at zero. All fluctuations move with the group velocity v_g to the right and reach the garage at a constant rate. After one time of flight, T, all correlations with the initial configuration have disappeared. The rounding in G at T is of order $\Delta t \sim T^{1/z}$, and is due to the broadening of the remaining correlation cones just before they are absorbed by the garage.

C. Inside the N phase

In the N phase, the fluctuations in the total number of cars on the road, $G(N_s, 0)$, are not proportional to N_s but are only of order one. The garage is not macroscopically occupied any more and acts very much like an ordinary road site. Figure 11 shows the behavior of $G(N_s, \tau)$ inside the N phase. The total number of cars in the system is conserved, such that G reduces to

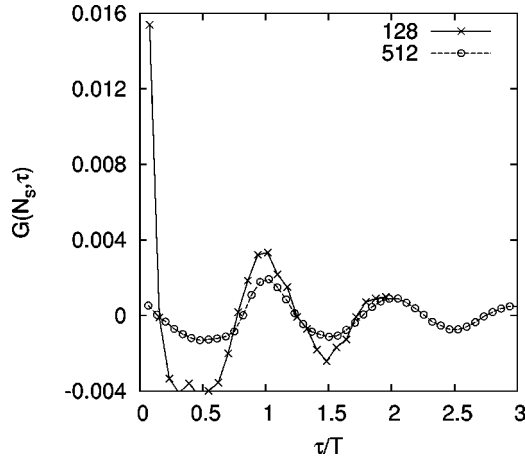$$G(N_s, \tau) = \langle N_p(t_0 + \tau) N_p(t_0) \rangle - \bar{N}_p^2, \qquad (24)$$

FIG. 11. The parked car correlator $G(N_s, \tau)$ in the N phase; the garage is no longer macroscopically occupied, such that G does not scale with N_s anymore. The correlations light up, like a lighthouse beam, at every time-of-flight interval $\tau_{\text{flight}} = T$. The data are obtained at $\rho_o = 0.25$ and $\alpha = 1$.

and behaves similar to the autocorrelator $g(0, \tau)$. The group velocity is nonzero, and therefore G decays exponentially fast. However, because of PBC, G comes back to live, like a lighthouse light beam, after every time-of-flight interval T ($= N_s / v_g$), with an amplitude of order $T^{2(\chi-1)/z}$ and with a temporal width of order $l \sim T^{1/z}$.

D. At the C-N transition

Figure 12 shows how $G(N_s, \tau)$ at the C-N transition decays in time for various system sizes. At small τ, it decays linearly, rather like in the C phase, but then it seems to oscillate with a period determined by the time-of-flight time scale T; $G(N_s, \tau)$ goes through zero at about $t \simeq \frac{1}{2} T$ and shows a strong anticorrelation at $t \simeq T$ (the maximum lies just before it).
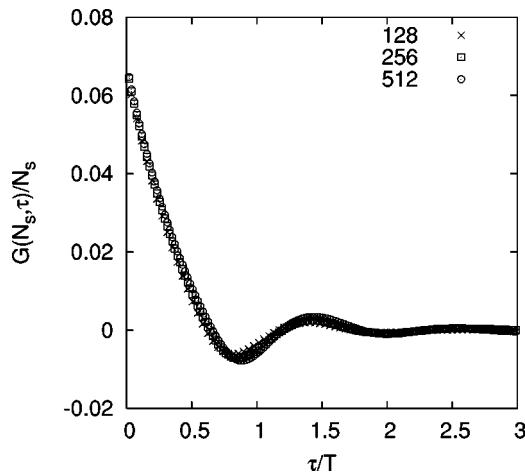

FIG. 12. The parked car correlator $G(N_s, \tau)$ at the C-N transition; it has a strong anticorrelation after one time-of-flight interval T, even though for small τ it decays linearly just like in the C phase (and with almost the same group velocity). The data are obtained at $\rho_o = \alpha = 0.25$.
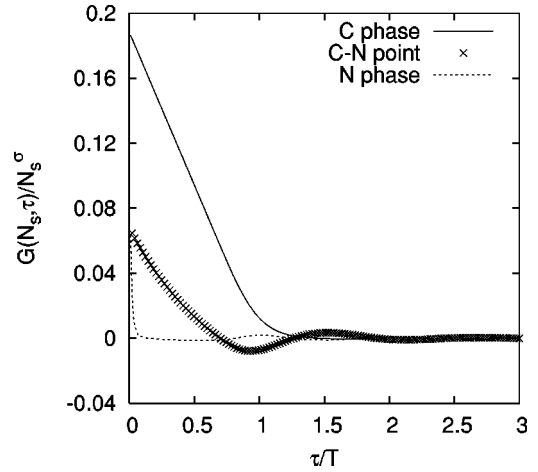

FIG. 13. Evolution of the parked car correlator $G(N_s, \tau)$ through the phase boundary at fixed system size $N_s = 512$ along a line of constant group velocity: $v_g = 1 - 2\rho_r$ with $\rho_r = 0.25$; $\sigma = 1$ in the C phase and at the C-N transition, while $\sigma = 0$ in the N phase.

Figure 13 illustrates how the transmission of information through the garage commences at the phase transition. Suppose we approach the transition point from the C phase following a line of constant α. In the C phase, these lines coincide with lines of constant group velocity. (In the N phase, v_g is constant along lines of constant ρ_o.) So nothing changes on the road until we hit the transition point, and G decays linearly to zero and remains zero after one time-of-flight time scale. At the transition point, G transforms abruptly into the oscillatory shape, with an anticorrelation after one time of flight. After that, it reduces inside the N phase to the lighthouse shape, in which $G(N_s, 0)$ oscillates in phase and does not scale with N_s any more.

The anticorrelations at the C-N transition point are intriguing and need to be explained. Imagine a localized positive density fluctuation at the beginning of the road at time t_0. In the MC phase, it simply sits there while broadening as $l \sim \tau^{1/z} \sim \tau^{2/3}$ and weakening in amplitude as $\tau^{2(\chi-1)/z} \sim \tau^{-2/3}$. In the C phase, it broadens and weakens in the same manner, but travels like a solitary wave to the right with velocity $v_g = 1 - 2\rho_r$ and drops out of the road after one time-of-flight unit T. In the N phase, it behaves very much the same, except that the positive density fluctuation creates a deficit inside the garage since the number of cars in the garage is finite, such that fewer cars can be put on the road in the immediate wake of the positive fluctuation. Therefore, in the N phase, every positive fluctuation carries a compensating (again localized) negative tail with it. Figure 14 illustrates the difference schematically, and our numerical simulations confirm this picture.

One could say that in the N phase positive and negative local excitations are bound in pairs, and that they unbind at the N-C transition. On approach of the transition from the N side, the width of the negative tail grows, but with conserved total area (equal to the area of the positive part of the excitation), because the average number of cars in the garage increases towards the transition (and diverges) and therefore the reduced car output is being spread over more time. At the

(a)

Parking Garage

$\Delta n(x)$

0

x

(b)

Parking Garage

$\Delta n(x)$
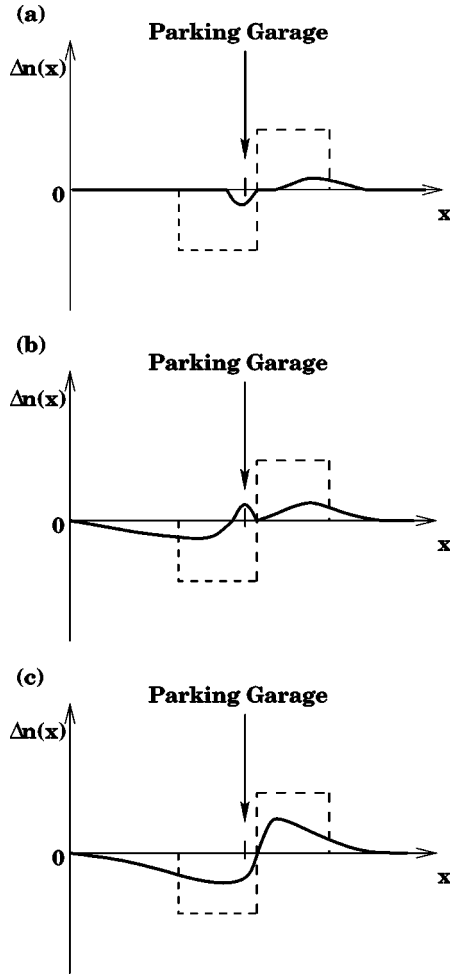
0

x

(c)

Parking Garage

$\Delta n(x)$

0

x

FIG. 14. Schematics of soliton-type local perturbations diffusively spreading from the initial configurations (dashed lines) to one time-of-flight interval later (solid lines): (a) in the C phase, (b) at the C-N transition, and (c) in the N phase.

transition point itself, the negative tail has vanished, except for the finite-size scaling effect of order $(N_s)^{-1/2}$.

Local excitations, thus, behave quite interestingly. However, they do not explain the difference in the behavior of G at the transition point and in the C phase. These solitary waves do not deplete the garage sufficiently to trigger its bottom because at the transition point the number of parked cars still diverges as $\sqrt{N_s}$.

Only nonlocal excitations, which encompass the entire system, are able to empty out the garage. Consider an excitation where the density of cars is globally and uniformly enhanced along the entire road, $\rho_i = \rho_r + \Delta$. This requires that the number of cars to be taken out of the garage should be proportional to the road length N_s. In the C phase, those will not deplete the garage because the number of parked cars is also proportional to N_s. Such a Δ regiment of extra cars marches with group velocity v_g to the right, reaches the end of the road, and thus returns to the garage at a rate uniformly in time, row by row. Throughout this process, the garage is not aware of the existence of the regiment since we did not hit its bottom and because the information cones $l \sim \tau^{1/z}$ on the road do not broaden fast enough compared to
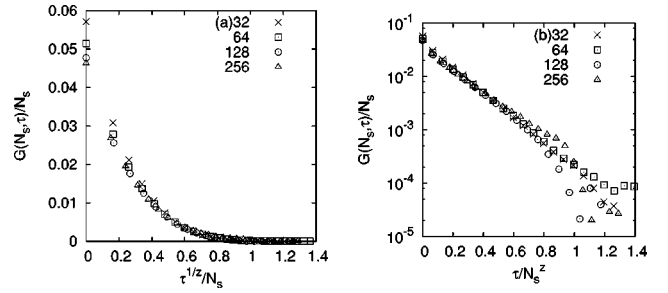

FIG. 15. The parked car correlator $G(N_s, \tau)$ at the MC-N transition; the shape is very similar to that inside the MC phase, see Fig. 9. The data are obtained at $\rho_o = 1/2$ and $\alpha = 1$.

the group velocity to maintain its memory at the garage exit. This implies that the exact stationary state rebuilds itself in the wake of the regiment, and also that the enhanced road density decays linearly in time and vanishes completely after one time of flight T, just like our G as a function of time τ.

At the transition point, the garage only contains $N_p \sim \sqrt{N_s}$ cars, so that the same type of global uniform excitation can only have an amplitude of order $(N_s)^{-1/2}$, and always depletes the garage. This regiment of cars travels to the right with velocity v_g, just like in the C phase, but in its wake the garage cannot rebuild the stationary state because it is empty. A depleted road density is established in the wake of the enhanced excitation, and thus anticorrelations build up, and after one time of flight the average density of cars on the road is below normal (and a surplus of cars resides in the garage). This explains qualitatively the oscillatory behavior of G at the transition point and the anticorrelations at T.

E. At the MC-N transition

The correlations at the MC-N transition are less spectacular than at the C-N transition. The group velocity is zero inside the MC phase, and still remains zero at the MC-N transition. Only inside the N phase, does it start to shift continuously away from zero. Figure 15 shows how G scales at the MC-N transition. These numerical results are almost the same as those inside the MC phase in Fig. 9.

Recall the argument about the behavior of $G(\tau, N_s)$ inside the MC phase (in section IX A), and imagine how this was modified at the MC-N transition. The factor N_s in Eq. (20) represents the number of cars on the road (the number of sites x_0 that are occupied). This should be modified to $\rho_o N_s - a\sqrt{N_s}$, since the number of parked cars scales as $\sqrt{N_s}$. The other terms, the spreading in time of the correlation cones and the autocorrelations on the road, are likely unchanged. Such differences are subtle and not surprisingly numerically invisible.

X. SUMMARY AND CONCLUSION

In this paper, we presented the scaling properties of dynamic condensate phase transitions in terms of an 1D asymmetric exclusion process with a parking garage. There are two types of condensate phases: the maximal current (MC) phase, where the road controls the density of cars on the road, and the condensate (C) phase, where the garage (as a

reservoir) controls the number of cars on the road. The existence of a group velocity is crucial for understanding the behaviors of correlations and the density profiles in these two phases and at the phase transitions.

At both condensate transitions, the number of parked cars scales as $N_p \sim N_s^{y_p}$ with $y_p = \frac{1}{2}$, while on approach of the transition, the density of parked cars vanishes linearly with the control parameters (the total density of cars in the system and the exit probability from the garage) $\rho_p \sim |\epsilon|^\beta$, with $\beta = 1$. Also, the transition points represent the onset of communication of information through the garage. This leads to interesting autocorrelations in the number of parked cars, particularly at the C-N transition, due to the nonzero group velocity and associated time-of-flight effects.

Our parking garage model is a bare-bone version of dynamic Bose condensation and of queuing phenomena, like traffic jams. The fundamental issue that needs further study is whether the above scaling behavior, in particular the values of the critical exponents, are universal or not. How do more realistic interactions between the cars on the road change this? Do the simplifications in the traffic jams, like a stationary truck versus a moving one and ignoring the spatial structure inside the queue of cars behind it by collapsing them (piling them up) into a "garage," affect the exponents?

There is some evidence suggesting that the exponents are indeed robust, e.g., in the Janowsky and Lebowitz [11] model, the same simple KPZ-type values of the exponents appear in the fluctuations of the queue, although a detailed study of the queuing transition itself needs still to be performed. In addition, introducing short-range car-car interactions [15,21] do not seem to change the exponents either [22].

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